International Journal of Theoretical Physics, Vol. 36, No. 9, 1997

Reducibility of Supersymmetric Quantum Mechanics

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Received December 17, 1996

We construct *deformed* annihilation and creation operators of the harmonic oscillator context in terms of the parity operator and realize in that way the superalgebra $sqm(2)$ of supersymmetric quantum mechanics. Moreover, this specific example is related to the physical application known as the Calogero problem. The reducibility of supersymmetric quantum mechanics is then established for arbitrary *odd* superpotentials, but not for even ones. We also get (minimal) dynamical algebras in both cases, shedding new light on such physical quantities as the Runge–Lenz vector.

1. INTRODUCTION

Since the early days of (classical as well as quantum) physics—let us say mechanics—one of the most interesting tools certainly is the study of the harmonic oscillator (Shankar, 1980; Capri, 1985). Its one-dimensional (spatial) version is already sufficient for considering extensions to realistic three- or d-dimensional applications. It has also been considered as a starting system for illustrating remarkable developments in quantum groups and algebras (Drinfeld, 1986; Jimbo, 1985; Macfarlane, 1989; Biedenharn, 1989).

Moreover, very recently, Palev and Stoilova (1994) have (re)visited the so-called three-dimensional *Wigner* oscillators (Wigner, 1950) by classifying their (purely bosonic) characteristics into three categories referring to three specific $\mathbb{Z}_2$-graded structures, i.e., the simple Lie superalgebras (Kac, 1977;

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0020-7748/97/0900-1991$12.50/0

Cornwell, 1989), respectively denoted by $osp(1|6)$, $sl(1|3)$, and $osp(3|2)$. We immediately mention that, in the one-dimensional context, these superalgebras $osp(1|6)$ and $osp(3|2)$ *both* reduce to $osp(1|2)$ and lead to the (already well-known study of) parabosonic oscillator (Ohnuki and Kamefuchi, 1982) of order p, while the superalgebra $sl(1|3)$ simply becomes $sl(1|1)$. This last structure reveals an unexpected property: $sl(1|1)$ is isomorphic to the Witten superalgebra (Witten, 1981) currently denoted by $sqm(2)$, which characterizes supersymmetric quantum mechanics (SSQM) when the two supercharges are identified with the annihilation and creation operators.

Let us also point out from another set of well-known properties that the above-mentioned parabosonic oscillator and its operators can be seen as (purely) generalized *deformed* bosonic ones: this directly comes out when the Chatuverdi and Srinivasan (1991) approach is related to deformations through the arguments developed very recently by Bonatsos and Daskaloyannis (1993).

It is thus natural to ask for the parallel possibility with regard to the third superalgebra $sl(1|1)$ [or $sqm(2)$ characterizing SSQM].

In fact, we want to show in this paper that we are able to construct annihilation and creation operators from generalized *deformation* requirements (Chaturvedi and Srinivasan, 1991) expressed in terms of the parity operator, an important discrete operator distinguishing the even or odd character of superpotentials, for example, in SSQM.

In that way, for general *odd* superpotentials (the harmonic oscillator one is only a particular example), we will show that SSQM is a *reducible* theory realized in terms of bosonic operators only. Such developments will also lead us to a very interesting physical application, known as the supersymmetric Calogero problem (d'Hoker *et al.*, 1989; Lahiri *et al.*, 1990; Cooper *et al.*, 1995; Celka and Hussin, 1987; Jayaraman and de Lima Rodrigues, 1994). In fact, this Calogero problem will be related to generalized *deformed* harmonic oscillator developments obtained through the introduction of the parity operator in the corresponding *supercharges*, the latter idea having been exploited in a different context by Gendenshtein and Krive (1985).

The contents of this paper are as follows. In Section 2, we realize the supersymmetric harmonic oscillator in terms of deformed purely bosonic operators appearing, in particular, as functions of the parity operator and acting on the Fock state vectors. In Section 3, we relate these developments to the physically interesting Calogero problem. In Section 4, we generalize the results of Section 2 to arbitrary *odd* superpotentials, but we also discuss the *even* context.

2. ON A GENERALIZED DEFORMED SUPERSYMMETRIC HARMONIC OSCILLATOR IN TERMS OF THE PARITY OPERATOR

In the one-dimensional context, the Wigner type II oscillator (Palev and Stoilova, 1994) admits the $sl(1|1)$-superalgebra characterized by the structure relations (in terms of commutators [.,.] and anticommutators {.,.})

$$A^2 = A^{\dagger 2} = 0 \tag{2.1a}$$

$$[\{A^\dagger, A\}, A] = 0 \tag{2.1b}$$

where capital letters refer to specific annihilation (A) and creation ($A^\dagger$) operators, distinguished from the usual bosonic ones (a and $a^\dagger$) characterized by the well-known relations

$$[a, a^\dagger] = 1 \tag{2.2}$$

$$H_B = \frac{1}{2}\{a, a^\dagger\}$$

where H_B is the expected bosonic Hamiltonian. If we define the *supersymmetric* Hamiltonian H_{SS} by

$$H_{SS} = \frac{1}{2}\{A, A^\dagger\} \tag{2.3}$$

we notice that the relations (2.1) and (2.3) generate a structure which is isomorphic to the Witten superalgebra *sqm* (2), where A and $A^\dagger$ play the role of the supercharges. Let us consider their action on a Fock basis $\{|n\rangle\}$ characterized by the occupation number $n = 0, 1, 2, \ldots$ appearing as eigenvalues of the occupation operator N. Following recent generalized deformation requirements (Chaturvedi and Srinivasan, 1991), we thus ask for

$$A|n\rangle = \sqrt{F(n)}|n-1\rangle$$

and (2.4)

$$A^\dagger|n\rangle = \sqrt{F(n+1)}|n+1\rangle$$

but with the extra condition

$$F(n)F(n-1) = 0 \tag{2.5}$$

in order to ensure the nilpotencies (2.1a). Simple considerations lead to two possible choices, hereafter denoted $F_\pm$, given by

$$F_\pm(n) = n[1 \pm (-1)^n] \tag{2.6}$$

where $(-1)^n$ naturally refer to the eigenvalues ± 1 of the parity operator P, the expressions (2.6) corresponding trivially to the operators

$$F_{\pm}(N) = N(1 \pm P) \tag{2.7}$$

These two possibilities lead to *nonequivalent* supersymmetric developments due to the (nonequivalent) sets of operators ensuring the superalgebra relations (2.1) and (2.3). In terms of the old bosonic operators $(a, a^{\dagger})$, we get the two sets

$$A_{\pm} = \frac{1}{\sqrt{2}}\, a(1 \mp P), \qquad A_{\pm}^{\dagger} = \frac{1}{\sqrt{2}}\, a^{\dagger}(1 \pm P) \tag{2.8}$$

where the signs have to be kept in correspondence. In the first $(+)$-context, it is easy to verify that the relations (2.3) and (2.4) become

$$H_{\text{SS},(+)} = \frac{1}{2}\,\{A_{+}, A_{+}^{\dagger}\} = \frac{1}{2}\,\{a, a^{\dagger}\} + \frac{1}{2} P \tag{2.9a}$$

$$A_{+}|n\rangle = \frac{1}{\sqrt{2}}\,[1 - (-1)^n]\,\sqrt{n}|n - 1\rangle \tag{2.9b}$$

$$A_{+}^{\dagger}|n\rangle = \frac{1}{\sqrt{2}}\,[1 + (-1)^n]\,\sqrt{n + 1}|n + 1\rangle \tag{2.9c}$$

This leads to the information

$$\begin{aligned} H_{\text{SS},(+)}|2n\rangle &= (2n + 1)|2n\rangle \\ H_{\text{SS},(+)}|2n + 1\rangle &= (2n + 1)|2n + 1\rangle \end{aligned} \tag{2.10}$$

so that we get in particular a *broken* supersymmetric model due to the result

$$H_{\text{SS},(+)}|0\rangle = |0\rangle \tag{2.11}$$

which will be discarded in the following. On the contrary, we will keep the second $(-)$-context because it leads to an *exact* (unbroken) supersymmetric model. We get

$$H_{\text{SS},(-)} = \frac{1}{2}\,\{A_{-}, A_{-}^{\dagger}\} = \frac{1}{2}\,\{a, a^{\dagger}\} - \frac{1}{2} P \tag{2.12a}$$

$$A_{-}|n\rangle = \frac{1}{\sqrt{2}}\,[1 + (-1)^n]\,\sqrt{n}|n - 1\rangle \tag{2.12b}$$

$$A_{-}^{\dagger}|n\rangle = \frac{1}{\sqrt{2}}\,[1 - (-1)^n]\,\sqrt{n + 1}|n + 1\rangle \tag{2.12c}$$

leading to the information

$$H_{SS,(-)}|2n\rangle = 2n|2n\rangle, \qquad H_{SS,(-)}|2n+1\rangle = (2n+2)|2n+1\rangle \tag{2.13}$$

and to the evident particular result on the vacuum state

$$H_{SS,(-)}|0\rangle = 0 \tag{2.14}$$

Both proposals (2.9) and (2.12) are realizations of the Witten superalgebra with generalized deformed bosonic generators only (without introducing Pauli matrices, for example).

We now point out the interest of the parity operator in Witten *matrix* form of SSQM characterized by the relations

$$Q^2 = Q^{\dagger 2} = 0, \qquad [Q, H_{W(H.O.)}] = [Q^\dagger, H_{W(H.O.)}] = 0 \tag{2.15a}$$

$$H_{W(H.O.)} = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}\{a, a^\dagger\} + \frac{1}{2}\sigma_3 \tag{2.15b}$$

when the supercharges are given in the harmonic oscillator case by

$$Q = \sqrt{2}\, a^\dagger \sigma_-, \qquad Q^\dagger = \sqrt{2}\, a\sigma_+ \tag{2.15c}$$

where σ_3, $\sigma_\pm = \sigma_1 \pm i\sigma_2$ are the usual Pauli matrices. Through the *unitary* transformation

$$U = \frac{1}{2}\begin{pmatrix} 1+P & -(1-P) \\ 1-P & 1+P \end{pmatrix} \tag{2.16}$$

expressed in terms of the above-mentioned parity operator, we get an equivalent representation given by

$$H'_W = \frac{1}{2}\{Q', Q'^\dagger\} = \begin{pmatrix} H_{SS,(+)} & 0 \\ 0 & H_{SS,(-)} \end{pmatrix} \tag{2.17a}$$

with the new supercharges

$$Q' = \begin{pmatrix} -A_+^\dagger & 0 \\ 0 & A_-^\dagger \end{pmatrix}, \qquad Q'^\dagger = \begin{pmatrix} -A_+ & 0 \\ 0 & A_- \end{pmatrix} \tag{2.17b}$$

We have thus constructed a completely *reducible* representation of the Witten superalgebra via the parity operator. The physical content is equivalent in both representations: only one bosonic state is associated with the energy eigenvalue $E = 0$ and twofold degeneracies (bosonic and fermionic states) are associated with nonzero energy eigenvalues. Let us only mention that the new fermionic number operator is given by

$$N'_F = U\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}U^{-1} = \frac{1}{2}\begin{pmatrix} 1 + P & 0 \\ 0 & 1 - P \end{pmatrix} \tag{2.18}$$

leading to fermionic states of the form

$$\begin{pmatrix} |2n\rangle \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ |2n + 1\rangle \end{pmatrix} \tag{2.19a}$$

and to bosonic states of the form

$$\begin{pmatrix} 0 \\ |2n\rangle \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} |2n + 1\rangle \\ 0 \end{pmatrix} \tag{2.19b}$$

3. A CONNECTION WITH THE CALOGERO SUPERPOTENTIAL

In the deformed harmonic oscillator case, fixed by the realizations (2.8) of the corresponding operators, let us choose the $(-)$-context leading to an exact supersymmetry according to the relations (2.12)–(2.14). In terms of the old bosonic position and momentum operators, denoted x and p as usual, we have that the super-Hamiltonian (2.12a) can be written

$$H_{SS,(-)} = \frac{1}{2}(p^2 + x^2) - \frac{1}{2}P = \frac{1}{2}(\hat{p}^2 + \hat{x}^2) \tag{3.1}$$

where the deformed $\hat{x}$ and $\hat{p}$ "variables" are defined by

$$\hat{x} = \frac{1}{\sqrt{2}}(x + ipP), \qquad \hat{p} = \frac{1}{\sqrt{2}}(p - ixP) \tag{3.2}$$

and satisfy

$$\{\hat{x}, \hat{p}\} = 0 \tag{3.3}$$

An interesting basis is immediately obtained by referring to the two subspaces of even (Ψ_+) and odd (Ψ_-) functions distinguished through the parity operator. We have

$$\Psi(x) = \frac{1}{2}(1 + P)\Psi(x) + \frac{1}{2}(1 - P)\Psi(x) \tag{3.4a}$$

$$= \Psi_+(x) + \Psi_-(x) \tag{3.4b}$$

and

$$H_{SS,(-)}\Psi_+(x) = n\Psi_+(x), \qquad H_{SS,(-)}\Psi_-(x) = (n + 1)\Psi_-(x) \tag{3.5}$$

so that the matrix formulation (2.17a) takes an interesting form here with the two-component states $(\Psi_+, \Psi_-)^\sim$. We thus have

$$\left[\frac{1}{2}(p^2 + x^2) - \frac{1}{2}\sigma_3\right]\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix} = \begin{pmatrix}n\Psi_+\\ (n + 1)\Psi_-\end{pmatrix} \tag{3.6}$$

We get only even energy eigenvalues ($n = 0, 2, 4, \ldots$) with well-defined states: for example, if $E = 0$, only the state $(|0\rangle, 0)^\sim$ is admissible; if $E = 2$, only the two states $(0, |1\rangle)^\sim$ and $(|2\rangle, 0)^\sim$ are eigenstates, etc. The reducibility of SSQM evidently appears in this context. If the parity eigenvalue is $(+1)$, the energy spectrum and the eigenfunctions are

$$E_n = 2n, \quad n = 0, 1, 2, \ldots, \qquad \Psi_n(x) = e^{-x^2/2}L_n^{-1/2}(x^2) \tag{3.7}$$

while if the parity eigenvalue is (-1), the energy spectrum and the eigenfunctions are

$$E_n = 2n + 2, \quad n = 0, 1, 2, \ldots, \qquad \Psi_n(x) = xe^{-x^2/2}L_n^{1/2}(x^2) \tag{3.8}$$

where $L_n^{\pm 1/2}(x^2)$ are the generalized Laguerre polynomials (Magnus *et al.*, 1966) very simply related to the well-known Hermite ones, such results being readily obtained from general information in SSQM.

These properties have directed our attention to the physically interesting Calogero problem (d'Hoker *et al.*, 1989; Lahiri *et al.*, 1990; Cooper *et al.*, 1995; Celka and Hussin, 1987; Jayaraman and de Lima Rodrigues, 1994) characterized by the supersymmetric Hamiltonian

$$H_{SS}^C = \frac{1}{2}(p^2 + x^2) - \lambda\left(1 - \frac{\lambda}{2x^2}\right) - \frac{1}{2}\sigma_3\left(1 + \frac{\lambda}{x^2}\right) \tag{3.9}$$

This solvable problem leads, if $\lambda \geq 1/2$, to the spectra and eigenfunctions

$$E_n = 2n, \quad n = 0, 1, 2, \ldots, \qquad \Psi_n(x) = x^\lambda e^{-x^2/2}L_n^{\lambda-1/2}(x^2) \tag{3.10}$$

and

$$E_n = 2n + 2, \quad n = 0, 1, 2, \ldots, \qquad \Psi_n(x) = x^{\lambda+1}e^{-x^2/2}L_n^{\lambda+1/2}(x^2) \tag{3.11}$$

corresponding to the eigenvalues $(+1)$ and (-1), respectively, of the Pauli matrix σ_3. The Calogero system and the deformed harmonic oscillator consequently have remarkable common properties (see also their connection if $\lambda = 0$).

Let us end this section by showing that, besides the property of having the same spectra, the above two systems also admit the same invariance orthosymplectic Lie superalgebra $osp(2|2)$ [already pointed out in the Calogero context by Balantekin (1985). This sheds light on a physical problem in terms of deformed characteristics (Drinfeld, 1986; Jimbo, 1985; Bonatsos and Daskaloyannis, 1993) relevant to quantum algebras. Indeed, using the same notations as in Balantekin (1985), let us mention here the ad hoc realization of the four *even* generators of $osp(2|2)$ for the Calogero problem on the forms

$$\begin{aligned} K_0 &= \frac{1}{4}\left(p^2 + x^2 + \frac{\lambda^2}{x^2} - \frac{\lambda}{x^2}\sigma_3\right) \\ K_+ &= \frac{1}{4}\left(p^2 - x^2 + 1 + 2ixp + \frac{\lambda^2}{x^2} - \frac{\lambda}{x^2}\sigma_3\right) \\ K_- &= \frac{1}{4}\left(p^2 - x^2 - 1 - 2ixp + \frac{\lambda^2}{x^2} - \frac{\lambda}{x^2}\sigma_3\right) \\ B &= -\frac{1}{4}\sigma_3 - \frac{\lambda}{2} \end{aligned} \tag{3.12a}$$

as well as that of the other four *odd* generators

$$\begin{aligned} V_+ &= \frac{1}{2}\left(p + ix + i\frac{\lambda}{x}\right)\sigma_-, \qquad V_- = \frac{1}{2}\left(p - ix + i\frac{\lambda}{x}\right)\sigma_- \\ W_+ &= \frac{1}{2}\left(p + ix - i\frac{\lambda}{x}\right)\sigma_+, \qquad W_- = \frac{1}{2}\left(p - ix - i\frac{\lambda}{x}\right)\sigma_+ \end{aligned} \tag{3.12b}$$

Such a realization corresponds to an *atypical* representation (Debergh, 1993) which is characterized through the Casimir operator of $osp(2|2)$ by

$$\tau = \frac{1}{4} + \frac{\lambda}{2} \qquad \text{if} \quad \sigma_3 = +1 \tag{3.13a}$$

and

$$\tau = \frac{3}{4} + \frac{\lambda}{2} \qquad \text{if} \quad \sigma_3 = -1 \tag{3.13b}$$

The parallel developments are relevant for the deformed harmonic oscillator, so that we get now corresponding to the super-Hamiltonian (3.1)

$$B' = -\frac{1}{4}P, \qquad K_0' = \frac{1}{4}(p^2 + x^2) \tag{3.14a}$$

$$K_+ = \frac{1}{4}(p^2 - x^2 + 1 + 2ixp), \qquad K_- = \frac{1}{4}(p^2 - x^2 - 1 - 2ixp)$$

and

$$V'_+ = \frac{1}{4}(p + ix)(1 + P), \qquad V_- = \frac{1}{4}(p - ix)(1 + P) \tag{3.14b}$$

$$W'_+ = \frac{1}{4}(p + ix)(1 - P), \qquad W_- = \frac{1}{4}(p - ix)(1 - P)$$

This is once again an atypical representation characterized by $\tau = 1/4$ and $\tau = 3/4$ according to the $(+1)$ and (-1) eigenvalues of the parity operator.

We see that the two super-Hamiltonians (3.1) and (3.9) have the same spectra and that the two systems admit the same invariance superalgebra, the unique difference lying in the fact that the representations of this invariance superalgebra are different (they evidently become identical when $\lambda = 0$, i.e., when the Calogero problem reduces to the harmonic oscillator case).

4. EXTENSION TO ARBITRARY SUPERPOTENTIALS

Let us generalize the supercharges (2.15c) as usual to the following forms given in terms of the superpotential $W(x)$:

$$Q = (p + iW(x))\sigma_-, \qquad Q^\dagger = (p - iW(x))\sigma_+ \tag{4.1}$$

and let us distinguish the parities of $W(x)$.

4.1. The Superpotential Is an Odd Function of x

The meaningful representation is immediately obtained through the unitary transformation (2.16). Adding a subscript $(\mathcal{O})$ referring to this *odd* context, we get

$$Q_{(\mathcal{O})} = UQU^{-1} = \frac{1}{2}\begin{pmatrix} -(p + iW(x))(1 + P) & 0 \\ 0 & (p + iW(x))(1 - P) \end{pmatrix} \tag{4.2a}$$

$$Q^\dagger_{(\mathcal{O})} = UQ^\dagger U^{-1} = \frac{1}{2}\begin{pmatrix} -(p - iW(x))(1 - P) & 0 \\ 0 & (p - iW(x))(1 + P) \end{pmatrix} \tag{4.2b}$$

and the super-Hamiltonian becomes a diagonal operator

$$H_{SS(\mathcal{O})} = \frac{1}{2}\begin{pmatrix} p^2 + W^2(x) + W'P & 0 \\ 0 & p^2 + W^2(x) - W'P \end{pmatrix}$$

$$= \frac{1}{2}(p^2 + W^2(x)) + \frac{1}{2}W'P\sigma_3 \tag{4.2c}$$

where W' refers to the first derivative of W. The results (4.2) once again show the complete reducibility of SSQM in the present context.

4.2. The Superpotential Is an Even Function of x

Characterizing this even context by subscript ($\mathscr{E}$), we easily get

$$Q_{(\mathscr{E})} = UQU^{-1} = \frac{1}{2}\begin{pmatrix} -p(1+P) & -iW(x)(1-P) \\ iW(x)(1+P) & p(1-P) \end{pmatrix} \tag{4.3a}$$

$$Q^{\dagger}_{(\mathscr{E})} = UQ^{\dagger}U^{-1} = \frac{1}{2}\begin{pmatrix} -p(1+P) & -iW(x)(1+P) \\ iW(x)(1-P) & p(1+P) \end{pmatrix} \tag{4.3b}$$

leading to the nondiagonal super-Hamiltonian

$$\begin{aligned} H_{\mathrm{SS}(\mathscr{E})} &= \frac{1}{2}\begin{pmatrix} p^2 + W^2(x) & W' \\ W' & p^2 + W^2(x) \end{pmatrix} \\ &= \frac{1}{2}(p^2 + W^2(x)) + \frac{1}{2}W'\sigma_1 \end{aligned} \tag{4.3c}$$

In contrast to the odd context, we see that the even case does not lead to a reducible representation. Note in addition that the Hamiltonian (4.3c) can be identified with the original Witten one

$$H_{\mathrm{W}} = \frac{1}{2}(p^2 + W^2) + \frac{1}{2}W'\sigma_3 \tag{4.4}$$

via the other unitary transformation

$$U'_{\mathrm{W}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4.5}$$

but leading to nondiagonal supercharges of the type

$$Q_{\mathrm{W}} = U'_{\mathrm{W}}Q_{(\mathscr{E})}U'_{\mathrm{W}} = \frac{1}{2}\begin{pmatrix} -(p - iW(x))P & -(p - iW(x)) \\ -(p + iW(x)) & -(p + iW(x))P \end{pmatrix} \tag{4.6}$$

In this *even* context of SSQM, the superalgebra *sqm*(2) is not reducible. In fact, we discover that *sqm*(2) does not characterize the applications with these even superpotentials, but has to be replaced by *sqm*(3) generated by the super-Hamiltonian (4.4) and *three* supercharges given, for example, by

$$
\begin{aligned}
Q_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & p - iW(x) \\ p + iW(x) & 0 \end{pmatrix} \\
Q_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i(p - iW(x)) \\ i(p + iW(x)) & 0 \end{pmatrix} \\
Q_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} -i(p - iW(x))P & 0 \\ 0 & -i(p + iW(x))P \end{pmatrix}
\end{aligned}
\tag{4.7}
$$

These three supercharges generate $sqm(3)$ characterized by the structure relations

$$
\begin{aligned}
&\{Q_1, Q_2\} = \{Q_1, Q_3\} = \{Q_2, Q_3\} = 0 \\
&Q_1^2 = Q_2^2 = Q_3^2 = \frac{1}{2}(p^2 + W^2(x)) + \frac{1}{2} W'\sigma_3 = H_W \equiv (3.4) \\
&[Q_k, H_W] = 0, \qquad k = 1, 2, 3
\end{aligned}
\tag{4.8}
$$

It also has to be noticed that these three supercharges satisfy the *commutation* relations

$$
[Q_j, Q_k] = 2i\epsilon_{jkl} H_W L_l \tag{4.9}
$$

where

$$
L_1 = \begin{pmatrix} 0 & iP \\ -iP & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.10}
$$

The relations (4.9) are then completed by

$$
[L_j, L_k] = 2i\epsilon_{jkl} L_l \tag{4.11a}
$$

and

$$
[L_j, Q_k] = 2i\epsilon_{jkl} Q_l \tag{4.11b}
$$

while

$$
[H_W, L_j] = 0, \qquad j = 1, 2, 3 \tag{4.12}
$$

The results (4.8), (4.9), (4.11), and (4.12) finally lead to $so(4)$ as an invariance algebra for H_W whatever W is. As an interesting physical example, let us mention that the *dynamical* algebra of the hydrogen atom (the corresponding superpotential (d'Hoker *et al.*, 1989) being invariant under the action of the parity operator) is recovered here by using supersymmetric developments improved by the parity operator. In particular, we can identify the three supercharges (4.7) as the three components of the Runge–Lenz vector, a result improving the one of Lyman and Aravind (1993), who took care, in a different approach, of two components Q_1 and Q_2 only [leading only with L_3 to the *so*(3)-algebra]. Thus, our third supercharge Q_3 achieves the total connection between supersymmetric quantum mechanics and the nonrelativistic hydrogen atom, due to the parity operator.

In conclusion, we have thus shown that, by taking account of the parity operator, the odd and even families of superpotentials lead to different Lie superalgebras [*sqm*(2) and *sqm*(3), respectively] resulting in corresponding different—minimal—dynamical algebras [exemplified by the *so*(3)-algebra in the special case of the odd harmonic oscillator context or by the *so*(4)-algebra in the special case of the even hydrogen atom one].

ACKNOWLEDGMENT

N.D. is Chercheur, Institut Interuniversitaire des Sciences Nucléaires, Brussels.

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